THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5540 Advanced Geometry 2016-2017
Supplementary Exercise 2

In the following exercises, we assume the axioms of incidence and betweenness.

1. Let A, B, C and D be four points. Prove that:
   (a) if $A * B * C$ and $B * C * D$, then $A * B * D$ and $A * C * D$;
   (b) if $A * B * D$ and $B * C * D$, then $A * B * C$ and $A * C * D$.
2. Prove that if A, B and C are three points on a line such that $A * C * B$, then $AC \cup CB = AB$ and $AC \cap CB = \{C\}$;
3. Let $\angle BAC$ be an angle and let D be a point lying in the interior of $\angle BAC$. Show that the every point of the ray r_{AD} except the point A lies in the interior of $\angle BAC$.
4. Show that the vertex of a ray is uniquely determined by the ray, i.e. there is one and only one vertex of a ray.
5. Let A be a point. Prove that there exist infinitely many lines such that each of them passes through the point A.

Lecturer's comment:

1. Recall the line separation property with another formulation:

   If B is a point and l is a line passing through the point B, then we can define a equivalence relation on the points of $l \backslash \{B\}$ such that $A \sim C$ if and only if the line segment AC does not contain B, i.e. $A * B * C$ is not true.

   Furthermore, there are only two equivalence classes, hence we say A and C are said to be on the same side of B if $A \sim C$, otherwise they are said to be on opposite side of B.

   (a) Considering the above equivalence relation with a fixed point B. By the assumption that $A * B * C$, we know that $A \nsim C$. Furthermore, by axiom **B3**, $B * C * D$ implies that $C * B * D$ is not true, i.e. $C \sim D$. Therefore, $A \nsim D$ and so $A * B * D$.
   Note that $A * B * C$ and $B * C * D$ imply $D * C * B$ and $C * B * A$ by axiom **B1**. By the above argument, we have $D * C * A$, as well as $A * C * D$ by axiom **B1** again.

   (b) Considering the above equivalence relation with a fixed point B. By the assumption that $A * B * D$, we know that $A \nsim D$. Furthermore, by axiom **B3**, $B * C * D$ implies that $C * B * D$ is not true, i.e. $C \sim D$. Therefore, $A \nsim C$ and so $A * B * C$.
   Then, by the above and (a), $A * B * C$ and $B * C * D$ implies $A * C * D$.

2. (i) Let $D \in AC \cup CB$, then D lies on AC or CB (but probably both).
   Assume $D \in AC = \{X : A * X * C\} \cup \{A, C\}$. If $D = A$, $D = A \in AB$. If $D = C$, by the assumption $A * C * B$, $D = C \in AB$. If $A * D * C$, by the assumption $A * C * B$ and question 1(b), we have $A * D * B$ and so $D \in AB$. Therefore, $D \in AB$ for all cases.

Similarly, if $D \in CB$, then we have $D \in AB$.

Hence, $AC \cup CB \subset AB$.

(ii) Let $D \in AB = \{X : A * X * B\} \cup \{A, B\}$. If $D = A$ or $D = B$, it is clear that $D \in AC \cup CB$.

Now, assume that $A * D * B$. We consider the three points A, D and C. By axiom **B3**, exactly one of the following is true: $A * D * C$, $A * C * D$ and $D * A * C$.

However, if $D * A * C$, by the assumption $A * C * B$ and question 1(a), we have $D * A * B$ which contradicts to the assumption that $A * D * B$ (by axiom **B3**). Therefore, we only need to consider the first two cases.

For the case $A * D * C$, it means that $D \in AC$ and so $D \in AC \cup CB$.

For the case $A * C * D$, by the assumption $A * D * B$ and question 1(b), we have $C * D * B$ which means $D \in CB$ and so $D \in AC \cup CB$.

Combining the above cases, we have $AB \subset AC \cup CB$.

Now, $AC \cup CB \subset AB$ and $AB \subset AC \cup CB$. Therefore, $AC \cup CB = AB$.

3. Recall the definition that the interior of $\angle BAC$ consists of all points E such that E and C are on the same side of the line l_{AB}, and E and B are on the same side of the line l_{AC}.

   Therefore, what we have to show is that if $E \in r_{AD}$ and $E \neq A, D$, then E satisfies the above conditions.

   By the line separation property and the definition of a ray, for any point $E \in r_{AD}$ and $E \neq A, D$, the line segment ED does not contain the point A. Note that if ED contains a point $X \in l_{AB}$, then $X \neq A$. Also, $A, X \in l_{ED} \cap l_{AB}$ which contradicts to axiom **I1**. Therefore, ED does not contain any point of the line l_{AB} and E and D are on the same side of the line l_{AB}. Furthermore, D and C are the same side of the line l_{AB}, so E and C are on the same side of the line l_{AB}. By similar argument, we can show that E and B are on the same side of the line l_{AC} and the result follows.

4. Let A and B be vertices of a ray r. We claim that we must have $A = B$.

   Suppose the contrary, let $A \neq B$. Consider A as a vertex of r, then $B \in r \backslash \{A\}$. By axiom **B2**, we can choose a point C on $r \backslash \{A\}$ such that $A * B * C$. Also B is also a vertex of r and A, C are points on r, then A and C should be on the same side of B. However, note that C is a point on the ray r, but C and A are on opposite side of B, which is a contradiction. Therefore, $A = B$.

5. Firstly, there exists a line l which does not contain A (why?). By axiom **I2**, there are two distinct points B_1 and B_2 on l.

   By axiom **B2**, there exists B_3 on l such that $B_1 * B_2 * B_3$. By using axiom **B3** repeatly, we show that there is an infinite sequence of points B_n so that $B_n * B_{n+1} * B_{n+2}$ for all natural numbers n. Then, we have an infinite sequence of lines l_{AB_n} which passes through A.

   (Think: Why $l_{AB_i} \neq l_{AB_j}$ for $i \neq j$?)